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Regular and Exponential Convergence of Difference Schemes for the Heat-Conduction Equation

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Abstract—The mathematical model for the heat-conduction equation has several special characteristic properties. In this paper, we examine the following property. By increasing time, the solution of the problem tends to the solution of the corresponding elliptic problem. Moreover, the convergence takes place without oscillation and the convergence rate in l_2-norm is the same as the convergence rate of the exponential function to zero.

Applying some numerical process, it is not less important to require the preservation of the discrete analogues of the basic qualitative properties of the continuous solution at certain fixed numerical solution (or at all of them). We introduce the (σ, θ)-method which is the generalization both of the well-known Galerkin linear finite element method and the finite difference method and formulate the conditions of the preservation of the regular and exponential convergence.

Keywords—Heat-conduction equation, Initial-boundary value problem, Eigenvalue, Regular convergence, Exponential convergence.

1. INTRODUCTION

In domain $\Omega = [0,1] \times \Re_0^+$, we consider the initial-boundary value problem for the heat-conduction equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f = Lw + f, \qquad x \in (0,1), \quad t > 0$$

with initial condition $w(x,0) = w_0(x)$ and first boundary conditions $w(0,t) = \mu_1$ and $w(1,t) = \mu_2$, respectively.

We assume that the given functions f and w_0 are sufficiently smooth and μ_1, μ_2 are constants. Then, there exists a unique solution $w : \Omega \mapsto \Re$ having several special characteristic properties, see, e.g., [1–3].

In this paper, we examine the following property. It is known that if f does not depend on t then, by letting $t \to \infty$, the function $w(x,t)$ tends to the solution of the corresponding elliptic problem.

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Moreover, the convergence takes place without oscillation and the convergence rate in l_2-norm is the same as the convergence rate of the function $\exp(-\lambda_1 t)$ to zero. Here, $-\lambda_1$ denotes the greatest (negative) eigenvalue of the elliptic operator L.

As usual, the continuous problem cannot be solved analytically. Therefore, we apply some numerical process which can be described in the following way. First, we semidiscretize the continuous problem with respect to the space variable and obtain a Cauchy problem for a system of ordinary differential equations. For the time integration of this problem, an easily implementable, so called θ-method is applied. For both methods the meshes are assumed to be uniform and they are characterized by the step-sizes h and τ, respectively. Consequently on the uniform mesh $\Omega_{h,\tau} \subset \Omega$ we define an approximation $w_{h,\tau}$ to the solution w. Obviously, the basic question is the convergence, that is, when refining h and τ, the sequence $w_{h,\tau}$ should be convergent to w in the sense of a certain norm. This problem is widely investigated in the literature, and as typical, the requirement of convergence results in conditions for the choice of the step-sizes h and τ, see, e.g., [4–6].

It is not less important to require the preservation of the discrete analogues of the basic qualitative properties of the continuous solution mentioned above at certain fixed numerical solution (or at all of them).

On a fixed mesh $\Omega_{h,\tau}$, the realization of the numerical process given above yields a one-step iteration process with respect to the unknown vectors $\mathbf{y}^j$, $j = 1, 2, \dots$. Here the components of the vectors $\mathbf{y}^j \in \Re^n$ denote the approximation to the solution $w(x, j\tau)$ at the points of the space-discretization with $h = 1/n$, $n \in \mathbb{N}$.

Let us denote the unit matrix in $\Re^{n\times n}$ by $\mathbf{E}$ and the uniformly continuant tridiagonal matrix tridiag$[-1, 2, -1]$ by $\mathbf{Q}$, respectively. The dependence of the matrices on the dimension n will not be denoted. We suppose that α_n and β_n are real numbers for which $q_n = \alpha_n + \beta_n$ are positive. We define the matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ as follows:

$$\mathbf{X}_1 = \mathbf{E} + \alpha_n \mathbf{Q}, \qquad \mathbf{X}_2 = \mathbf{E} - \beta_n \mathbf{Q}. \tag{1}$$

A number of the numerical methods leads to the sequence of the iterations of the form

$$\mathbf{X}_1 \mathbf{y}_n^{j+1} = \mathbf{X}_2 \mathbf{y}_n^j + \tilde{\mathbf{c}}_n, \qquad j = 0, 1, 2, \dots, \tag{2}$$

where $n = 1, 2, 3, \dots$. For all n, $\tilde{\mathbf{c}}_n$ and $\mathbf{y}_n^0$ are given vectors. Throughout the paper, we refer to such a discretization method with fixed n as to an (α_n, β_n)-method. The notation (α, β)-method is used for the family of (α_n, β_n)-methods where $n \in \mathbb{N}$.

Obviously, the iteration process (2) is equivalent to the iteration

$$\mathbf{y}_n^{j+1} = \mathbf{X} \mathbf{y}_n^j + \mathbf{c}_n, \qquad j = 0, 1, 2, \dots, \tag{3}$$

where the notations

$$\mathbf{X} = \mathbf{X}_1^{-1} \mathbf{X}_2, \qquad \mathbf{c}_n = \mathbf{X}_1^{-1} \tilde{\mathbf{c}}_n$$

are used. Introducing the notations $\lambda_i^{(n)}$ and $\Lambda_i^{(n)}$, $i = 1, 2, \dots, n$ for the eigenvalues of matrices $\mathbf{Q}$ and $\mathbf{X}$, respectively, one has

$$\lambda_i^{(n)} = 4 \sin^2 \frac{i\pi}{2(n+1)}, \qquad \Lambda_i^{(n)} = 1 - q_n \frac{\lambda_i^{(n)}}{1 + \alpha_n \lambda_i^{(n)}}. \tag{4}$$

By the examination of the eigenvalues $\Lambda_i^{(n)}$, we can easily obtain the condition of the convergence (in l_2-norm) of the iteration process (2) to the limit vector

$$\mathbf{y}_n^* = (\mathbf{E} - \mathbf{X})^{-1} \mathbf{c}_n = \frac{1}{q_n} \mathbf{Q}^{-1} \tilde{\mathbf{c}}_n. \tag{5}$$

Namely, condition

$$\beta_n - \alpha_n < \frac{2}{\lambda_n^{(n)}} \tag{6}$$

must hold.

2. REGULAR CONVERGENCE OF THE (α,β)-METHOD

Assume that an (α_n,β_n)-method is convergent. Then, we have the estimation

$$\left\|\mathbf{y}_n^j - \mathbf{y}_n^*\right\| \le (\varrho_n)^j \left\|\mathbf{y}_n^0 - \mathbf{y}_n^*\right\|, \tag{7}$$

where $\varrho_n = \max(|\Lambda_1^{(n)}|, |\Lambda_n^{(n)}|)$.

DEFINITION 1. *A convergent (α_n,β_n)-method is said to be n-regularly convergent if*

(a) $\Lambda_1^{(1)} > 0$, *when* $n = 1$,
(b) *(7) is valid with* $\varrho_n = \Lambda_1^{(n)} > |\Lambda_n^{(n)}|$, *when* $n \ge 2$.

Shortly, $(\alpha_n,\beta_n) \in R(n)$.

DEFINITION 2. *If $(\alpha_n,\beta_n) \in R(n)$ for all $n = 1,2,\dots$, then, it is called uniformly regularly convergent.*

Shortly, $(\alpha_n,\beta_n) \in R(\mathbb{N})$.

If $(\alpha_n,\beta_n) \in R(n)$, then the sequence of i^{th} components $\{(\mathbf{y}_n^j - \mathbf{y}_n^*)_i,\ j \in \mathbb{N}\}$ tends to zero with at most finite number of sign-changes for every $i = 1,2,\dots,n$.

Clearly, $(\alpha_n,\beta_n) \in R(n)$ if and only if it is convergent and the condition

$$\frac{1-\beta_n\lambda_1^{(n)}}{1+\alpha_n\lambda_1^{(n)}} > \left|\frac{1-\beta_n\lambda_n^{(n)}}{1+\alpha_n\lambda_n^{(n)}}\right| = \frac{\left|1-\beta_n\lambda_n^{(n)}\right|}{1+\alpha_n\lambda_n^{(n)}} \tag{8}$$

is fulfilled. Due to the definition, $\Lambda_1^{(n)} > 0$ if and only if

$$\beta_n < \frac{1}{\lambda_1^{(n)}}. \tag{9}$$

If $\beta_n \le 1/\lambda_n^{(n)}$, then (8) is always true for our methods in question. If $\beta_n \in (1/\lambda_n^{(n)},\ 1/\lambda_1^{(n)})$, then, by the help of the relations

$$\lambda_1^{(n)} + \lambda_n^{(n)} = 4, \qquad \lambda_1^{(n)}\lambda_n^{(n)} = 4\sin^2\frac{\pi}{n+1},$$

(8) results in the condition

$$-4\sin^2\left(\frac{\pi}{n+1}\right)\alpha_n\beta_n + 2(\alpha_n - \beta_n) + 1 > 0. \tag{10}$$

Combining these conditions with the requirements of the convergence, we obtain the following proposition.

PROPOSITION 1. *Assume that $\alpha_n + \beta_n > 0$. Then, the inclusion $(\alpha_n,\beta_n) \in R(n)$ holds if and only if $\beta_n - \alpha_n < 2/\lambda_n^{(n)}$ and one of the conditions*

(1) $\beta_n \le 1/\lambda_n^{(n)}$,
(2) $\beta_n \in (1/\lambda_n^{(n)}, 1/\lambda_1^{(n)})$ *and (10) is fulfilled.*

REMARK. If $\alpha_n \le 0$ and $\beta_n \ge 0$, then $(\alpha_n,\beta_n) \in R(n)$ if and only if the method is convergent and condition (9) holds.

Let us examine the conditions of regular convergence by varying n. It is true that $\lim_{n\to\infty} 1/\lambda_1^{(n)} = \infty$, $\lim_{n\to\infty} 1/\lambda_n^{(n)} = 1/4$ and the convergence is monotone. Moreover, if $\beta_n - \alpha_n < 1/2$, then (10) holds for sufficiently large values of n. Combining the result with the condition of the convergence, we arrive at the following proposition.

PROPOSITION 2. *Assume that $\alpha_n + \beta_n > 0$ for all $n = 1, 2, \ldots$. If $\beta_n \leq 1/4$ for all $n = 1, 2, \ldots$, then $(\alpha, \beta) \in R(\mathbb{N})$. Moreover, if $\alpha_n \leq 0$ and $\beta_n < 1/2$, then $(\alpha, \beta) \in R(\mathbb{N})$.*

REMARK. If there exists a number $n_0 \in \mathbb{N}$ such that (α_n, β_n) is a convergent method and (9) holds for all $n \geq n_0$, then $(\alpha_n, \beta_n) \in R(n)$ for all $n \geq n_1$, where $n_1 \geq n_0$ is a well defined number.

Let us assume that $\alpha_n = \alpha$ and $\beta_n = \beta$ are constants. As a consequence of Proposition 2, we have that for any convergent (α, β)-method there exists a number n_0 such that $(\alpha, \beta) \in R(n)$ for all $n \geq n_0$.

If $\alpha \leq 0$, then $(\alpha, \beta) \in R(\mathbb{N})$ if and only if $\beta < 1/2$.

Assume now that both α and β are nonnegative. We examine the condition of the inclusion $(\alpha, \beta) \in R(\mathbb{N})$. Clearly, condition (9) holds for any $n = 1, 2, \ldots$ if and only if $\beta < 1/2$. Since the sequence $\{\alpha\beta \sin^2(\pi/(n+1))\}$ is monotonically decreasing, therefore it is sufficient to check (10) only for $n = 1$. It results in the condition

$$(1 + 2\alpha)(1 - 2\beta) > 0.$$

Since it is satisfied under our assumptions, we have the following proposition.

PROPOSITION 3. *Assume that $\alpha_n = \alpha$ and $\beta_n = \beta \geq 0$ are constants such that $\alpha + \beta > 0$. Then, $(\alpha, \beta) \in R(\mathbb{N})$ if and only if $\beta < 1/2$.*

3. EXPONENTIAL CONVERGENCE OF THE (α, β)-METHOD

Besides being regularly convergent, i.e., having a linear convergence to the limit vector, an (α_n, β_n)-method usually should also have an exponential convergence of the form

$$\left\|\mathbf{y}_n^j\right\| \leq \exp\left[-\lambda_1^{(n)} (\alpha_n + \beta_n) j\right] \left\|\mathbf{y}_n^0\right\|, \tag{11}$$

for all $j \in \mathbb{N}$.

DEFINITION 3. *An (α_n, β_n)-method is said to be n-exponentially convergent if it belongs to $R(n)$ and (11) holds.*

Shortly, $(\alpha_n, \beta_n) \in E(n)$.

DEFINITION 4. *If $(\alpha_n, \beta_n) \in E(n)$ for all $n = 1, 2, \ldots$, then it is called uniformly exponentially convergent.*

Shortly, $(\alpha, \beta) \in E(\mathbb{N})$.

It is clear that $(\alpha_n, \beta_n) \in E(n)$, if and only if $(\alpha_n, \beta_n) \in R(n)$ and

$$\Lambda_1^{(n)} \leq \exp\left[-\lambda_1^{(n)} (\alpha_n + \beta_n)\right]. \tag{12}$$

Let us represent $\Lambda_1^{(n)}$ in the form

$$\Lambda_1^{(n)} = \exp\left[-\lambda_1^{(n)} (\alpha_n + \beta_n) + p_n\right]. \tag{13}$$

In order to get (12), we have to prove that $p_n \leq 0$. Since (13) implies

$$p_n = \lambda_1^{(n)}(\alpha_n + \beta_n) + \ln\left(1 - \beta_n\lambda_1^{(n)}\right) - \ln\left(1 + \alpha_n\lambda_1^{(n)}\right),$$

therefore, using the power series expansion of the function $\ln(1 + x)$, we get that $p_n \leq 0$ if and only if the relation

$$\frac{1}{2!}\left(\beta_n^2 - \alpha_n^2\right) + \frac{\lambda_1^{(n)}}{3!}\left(\beta_n^3 - \alpha_n^3\right) + \frac{\left(\lambda_1^{(n)}\right)^2}{4!}\left(\beta_n^4 - \alpha_n^4\right) + \cdots \geq 0 \tag{14}$$

is valid. It yields the condition

$$\exp\left(\beta_n\lambda_1^{(n)}\right) - \exp\left(\alpha_n\lambda_1^{(n)}\right) + \lambda_1^{(n)}(\alpha_n - \beta_n) \geq 0. \tag{15}$$

So, for some fixed n, $(\alpha_n, \beta_n) \in R(n)$ implies that $(\alpha_n, \beta_n) \in E(n)$ if and only if (15) holds. Let us define a function g by $g(x) = \exp(x) - x$. Clearly, g is monotonically decreasing for $x < 0$ and monotonically increasing for $x > 0$, respectively. Moreover, $g(x) \geq g(-x)$ for $x \geq 0$. So, in case $\beta_n \geq |\alpha_n|$, (15) holds for any $n \in \mathbb{N}$. On the other hand, from (14), using condition $\alpha_n + \beta_n > 0$ and the properties of the function g, one can see that this condition is necessary, too. Namely, (14) implies the necessary condition $|\beta_n| \geq |\alpha_n|$. If $\beta_n > 0$, then we obtain the condition. If $\beta_n \leq 0$, then $\alpha_n > -\beta_n$, and $g(\alpha_n) > g(-\beta_n) \geq g(\beta_n)$, which yields a contradiction.

By the use of Propositions 1 and 2, we can summarize our results as follows.

PROPOSITION 4. *Assume that $\alpha_n + \beta_n > 0$ for all $n = 1, 2, \dots$. If $|\alpha_n| \leq \beta_n \leq 1/4$ for all $n = 1, 2, \dots$, then $(\alpha, \beta) \in E(\mathbb{N})$. Moreover, if there exists a number $n_0 \in \mathbb{N}$ such that $(\alpha_n, \beta_n) \in R(n)$ and $\beta_n \geq \max(|\alpha_n|, 1/4)$ hold for all $n \geq n_0$, then $(\alpha_n, \beta_n) \in E(n)$ for all $n \geq n_1$ where $n_1 \geq n_0$ is a well-defined number.*

PROPOSITION 5. *Assume that $\alpha_n = \alpha$ and $\beta_n = \beta \geq 0$ are constants such that $\alpha + \beta > 0$. Then, $(\alpha, \beta) \in E(\mathbb{N})$ if and only if $\beta \geq |\alpha|$ and $\beta < 1/2$.*

4. REGULAR AND EXPONENTIAL CONVERGENCE OF THE SPECIAL DIFFERENCE SCHEMES

Let us apply our statements choosing

$$\alpha_n = -\sigma + \theta q_n, \qquad \beta_n = \sigma + (1-\theta)q_n, \tag{16}$$

where $\sigma \in [0, 1/4)$ and $\theta \in [0, 1]$ are given numbers. A method with this choice will be called and denoted as a (σ, θ)-method. Obviously, when $\sigma = 0$ or $\sigma = 1/6$ we obtain the well-known finite difference method and linear finite element method, respectively.

Then, the theorem in question for the regular convergence can be stated as follows.

THEOREM 1. *If the condition*

$$q_n \leq \frac{1 - 4\sigma}{4(1-\theta)} \tag{17}$$

holds for all $n \geq 1, 2, \dots$, then the (σ, θ)-method is uniformly regularly convergent. Moreover, when $q_n = q$ is fixed, then $(\sigma, \theta) \in R(\mathbb{N})$ if and only if the condition

$$q < \frac{1 - 2\sigma}{2(1-\theta)} \tag{18}$$

is satisfied.

Hence, for the special schemes we can formulate the conditions of the uniform regularity.

(a) *Finite difference method* ($\sigma = 0$).
If

$$q_n \leq \frac{1}{4(1-\theta)} \tag{19}$$

holds for all $n = 1, 2, \dots$, then $(0, \theta) \in R(\mathbb{N})$. Moreover, when $q_n = q$ is fixed, then $(0, \theta) \in R(\mathbb{N})$ if and only if condition

$$q < \frac{1}{2(1-\theta)} \tag{20}$$

is satisfied.

(b) *Finite element method with linear elements* ($\sigma = 1/6$).
If
$$q_n \le \frac{1}{12(1-\theta)} \tag{21}$$
holds for all $n = 1, 2, \dots$, then $(1/6, \theta) \in R(\mathbb{N})$. Moreover, when $q_n = q$ is fixed, then $(1/6, \theta) \in R(\mathbb{N})$ if and only if condition
$$q < \frac{1}{3(1-\theta)} \tag{22}$$
is satisfied.

On the basis of Propositions 4 and 5 and by the use of (16) for the exponential convergence we have the following theorem.

THEOREM 2. *If*
$$q_n \le \begin{cases} \dfrac{1-4\sigma}{4(1-\theta)}, & \text{when } \theta \in \left[0, \dfrac{1}{2} + 2\sigma\right], \\ \dfrac{2\sigma}{2\theta - 1}, & \text{when } \theta \in \left(\dfrac{1}{2} + 2\sigma, 1\right], \end{cases} \tag{23}$$
then (σ, θ)-method is uniformly exponentially convergent. Moreover, when $q_n = q$ is fixed then $(\sigma, \theta) \in E(\mathbb{N})$ if and only if the condition
$$q < \begin{cases} \dfrac{1-2\sigma}{2(1-\theta)}, & \text{when } \theta \in \left[0, \dfrac{1}{2} + \sigma\right], \\ \dfrac{2\sigma}{2\theta - 1}, & \text{when } \theta \in \left(\dfrac{1}{2} + \sigma, 1\right] \end{cases} \tag{24}$$
holds.

Hence, we can formulate the conditions of the uniform exponential convergence of the special schemes.

(a) *Finite difference method* ($\sigma = 0$).
If
$$q_n \le \frac{1}{4(1-\theta)}, \qquad \theta \in \left[0, \frac{1}{2}\right] \tag{25}$$
holds for all $n \in \mathbb{N}$, then $(0, \theta) \in E(\mathbb{N})$. Moreover, when $q_n = q$ is fixed, then $(0, \theta) \in E(\mathbb{N})$ if and only if the condition
$$q < \frac{1}{2(1-\theta)}, \qquad \theta \in \left[0, \frac{1}{2}\right] \tag{26}$$
is satisfied.

(b) *Finite element method with linear elements* ($\sigma = 1/6$).
If
$$q_n \le \begin{cases} \dfrac{1}{12(1-\theta)}, & \text{when } \theta \in \left[0, \dfrac{5}{6}\right], \\ \dfrac{1}{3(2\theta - 1)}, & \text{when } \theta \in \left(\dfrac{5}{6}, 1\right] \end{cases} \tag{27}$$
holds for all $n \in \mathbb{N}$, then $(1/6, \theta) \in E(\mathbb{N})$. Moreover, when $q_n = q$ is fixed, then $(1/6, \theta) \in E(\mathbb{N})$ if and only if the condition
$$q < \begin{cases} \dfrac{1}{3(1-\theta)}, & \text{when } \theta \in \left[0, \dfrac{2}{3}\right], \\ \dfrac{1}{3(2\theta - 1)}, & \text{when } \theta \in \left(\dfrac{2}{3}, 1\right] \end{cases} \tag{28}$$
is satisfied.

REMARK. The conditions of the preservation of other qualitative properties like conservation of nonnegativity, monotonicity, and concavity of the initial function, are investigated in [1,7,8].

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